

COVERINGS OF TORUS KNOTS IN $S^2 \times S^1$ AND UNIVERSALS

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ABSTRACT. Let $t_{\alpha,\beta} \subset S^2 \times S^1$ be an ordinary fiber of a Seifert fibering of $S^2 \times S^1$ with two exceptional fibers of order α . We show that any Seifert manifold with Euler number zero is a branched covering of $S^2 \times S^1$ with branching $t_{\alpha,\beta}$ if $\alpha \geq 3$.

We compute the Seifert invariants of the Abelian covers of $S^2 \times S^1$ branched along a $t_{\alpha,\beta}$.

We also show that $t_{2,1}$, a non-trivial torus knot in $S^2 \times S^1$, is not universal.

1. INTRODUCTION.

The study of branched coverings of Seifert manifolds branched along fibers is an interesting project ([4]). These kind of coverings are also Seifert manifolds. In particular the problem of determining which Seifert manifolds are branched coverings with branching along a torus knot is interesting, where a *torus knot in M* is an ordinary fiber of a Seifert fibering of M .

Recall that in a branched covering of Seifert manifolds, the Euler number of the covering is a non-zero rational multiple of the Euler number of the target. Therefore in a branched covering $M \rightarrow N$ between Seifert manifolds, if the Euler number $e(N) \neq 0$, then $e(M) \neq 0$. In [3], a converse of this implication is proven for torus knots in the 3-sphere: If M is an orientable Seifert manifold with orientable basis and Euler number $e(M) \neq 0$, and $\tau_{p,q} \subset S^3$ is a torus knot with $2 \leq q < p$, then there is a branched covering $\varphi : M \rightarrow (S^3, \tau_{p,q})$.

What about if $e(M) = 0$? In this paper we answer this question: If M is an orientable Seifert manifold with orientable basis and Euler number $e(M) = 0$, and $t_{\alpha,\beta} \subset S^2 \times S^1$ is a torus knot (see Section 2.2.3),

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with $3 \leq \beta \leq \alpha/2$, then there is a branched covering $\varphi : M \rightarrow (S^2 \times S^1, t_{\alpha, \beta})$.

Resuming, all orientable Seifert manifolds with orientable basis can be constructed as branched coverings branched along torus knots in S^3 or in $S^2 \times S^1$.

In contrast with the result in [3], that all non-trivial torus knots in S^3 are universal in the sense above, we show that $t_{2,1} \subset S^2 \times S^1$ is a non-trivial torus knot which is not universal (Theorem 4.7).

This paper is organized as follows.

In Section 2 we give some definitions and list some useful results.

In Section 3 we give a list of the Abelian coverings of $(S^2 \times S^1, t_{\alpha, \beta})$. It follows that if $\varphi : M \rightarrow (S^2 \times S^1, t_{\alpha, \beta})$ is an Abelian branched covering and $H_1(M) \cong \mathbb{Z}$, then $M \cong S^2 \times S^1$, and φ is an unbranched cyclic covering space. Compare with the analogous result: If M is an n -fold cyclic branched covering of $(S^3, \tau_{p,q})$, then $H_1(M) = 0$ if and only if $(n, pq) = 1$.

In Section 4 we prove the main theorem of the paper (Corollary 4.6).

2. PRELIMINARIES.

Let S_n be the symmetric group in n symbols. If $G \leq S_n$, and $i \in \{1, \dots, n\}$, write $St_G(i) = \{\sigma \in G : \sigma(i) = i\}$; also write $St(i) = St_G(i)$ when $G = S_n$. We write $(1) \in S_n$ for the identity permutation.

A *branched covering* between two n -manifolds M and N is an open, proper map $\varphi : M \rightarrow N$ which is finite-to-one. To check that an open map $\varphi : M \rightarrow N$ is a branched covering, one finds a subcomplex $K \subset M$ of codimension two such that the restriction $\varphi| : M - \varphi^{-1}(K) \rightarrow N - K$ is a finite covering space. The subcomplex K , usually a submanifold, is called the *branching* of φ . The covering space $\varphi| : M - \varphi^{-1}(K) \rightarrow N - K$ is called the *associated covering space* of φ . The associated covering determines the branched covering (see [1]).

A covering space of n sheets $\psi : X \rightarrow Y$ defines a representation $\omega_\psi : \pi_1(Y) \rightarrow S_n$ as follows: Number the elements of the preimage of a base point $\psi^{-1}(*) = \{1, 2, \dots, n\}$, and for a class $[\alpha] \in \pi_1(Y)$, consider the liftings of α , $\alpha_1, \alpha_2, \dots, \alpha_n : I \rightarrow X$ where α_i starts in the point $i \in \psi^{-1}(*)$; then define $\omega_\psi([\alpha])(i) = \alpha_i(1)$, for $i = 1, 2, \dots, n$.

A homomorphism $\omega : \pi_1(Y) \rightarrow S_n$ determines a covering space of n sheets $\psi_\omega : X \rightarrow Y$, namely, the covering space corresponding to the

subgroup $\omega^{-1}(St(1)) \leq \pi_1(Y)$; this covering space can be completed into a branched covering $\psi_\omega : \bar{X} \rightarrow \bar{Y}$ ([1]). A representation of ψ , as ω_ψ above, is conjugate to the homomorphism ω (it only depends on the numbering of $\psi^{-1}(*)$).

We will describe a branched covering of a manifold M by giving a codimension two submanifold $K \subset M$ together with a representation $\omega : \pi_1(M - K) \rightarrow S_n$. For short, we write $N \rightarrow (M, K)$ for a branched covering of M branched along K .

2.1. Seifert manifolds. Let $\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_t, \beta_t$ be integers such that $\alpha_i > 0$ and $(\alpha_i, \beta_i) = 1$ for $i = 1, 2, \dots, t$. The *Seifert manifold* M associated to the *Seifert symbol* $(Oo, g; \beta_1/\alpha_1, \beta_2/\alpha_2, \dots, \beta_t/\alpha_t)$ is assembled as follows:

Let F be an oriented closed surface of genus g , and let $D_1, D_2, \dots, D_t \subset F$ be t disjoint 2-disks. Write $F_0 = \overline{F - \bigcup D_i}$, and $M_0 = F_0 \times S^1$. If $\partial F = q_1 \sqcup q_2 \sqcup \dots \sqcup q_t$ and $h = \{x\} \times S^1$ for some $x \in F_0$, we let $m_i \subset q_i \times S^1$ be a simple closed curve such that $m_i \sim q_i^{\alpha_i} h^{\beta_i}$. Let V_1, V_2, \dots, V_t be solid tori with meridians $\mu_1, \mu_2, \dots, \mu_t$, respectively, and let $\eta_i : \partial V_i \rightarrow q_i \times S^1$ be a homeomorphism such that $\eta_i(\mu_i) = m_i$ for $i = 1, 2, \dots, t$. Then the Seifert manifold M associated to the symbol $(Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$ is

$$M = M_0 \bigcup_{\cup \eta_i} \left(\bigcup_{i=1}^t V_i \right).$$

The circles $\{x\} \times S^1$ for $x \in F_0$ are called the *ordinary fibers* of M and the core e_i of V_i is called an *exceptional fiber of order α_i* if $\alpha_i > 1$; otherwise e_i is also an ordinary fiber. In any case e_i is called *the fiber of the ratio β_i/α_i* ($i = 1, 2, \dots, t$). The surface F is called the *orbit surface* or the *base* of M . Note that collapsing each fiber of M into a point gives an identification $p : M \rightarrow F$.

The manifold M associated to $(Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$ can be recovered unambiguously from $M_0 = F_0 \times S^1$ and the curves $m_i = q_i^{\alpha_i} h^{\beta_i}$, $i = 1, 2, \dots, t$. We call the pair $(F_0 \times S^1, \{m_i\}_{i=1}^t)$ a *frame* for $(Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$.

Let V be a solid torus. Any covering space of ∂V extends into a branched covering of V branched at most along the core of V . Therefore to describe a branched covering of $(Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$ branched along fibers, it suffices to construct a covering space of a frame for $(Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t, 0/1, \dots, 0/1)$.

The classification of Seifert manifolds is given by

Theorem 2.1 ([2]). *Two Seifert symbols represent homeomorphic Seifert manifolds by a fiber preserving homeomorphism if and only if one of the symbols can be changed into the other by a finite sequence of the following moves:*

0. *Permute the ratios.*
1. *Add or delete $\frac{0}{1}$.*
2. *Replace the pair $\frac{\beta_i}{\alpha_i}, \frac{\beta_j}{\alpha_j}$ by $\frac{\beta_i + k\alpha_i}{\alpha_i}, \frac{\beta_j - k\alpha_j}{\alpha_j}$.*

□

If $M = (Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$, then the number $e(M) = -\sum \frac{\beta_i}{\alpha_i}$ is called the *Euler number* of M .

Theorem 2.2 ([2]). *Let M and M' be Seifert manifolds with base spaces F and F' , respectively, and let $f : M \rightarrow M'$ be an orientation preserving and fiber preserving map. If the degree of the induced map on a typical fiber is n , and the degree of the induced map $\bar{f} : F \rightarrow F'$ is m , then $e(M) = (m/n)e(M')$.*

□

In particular if $\varphi : M \rightarrow M'$ is a branched covering branched along fibers of M' , $e(M') = 0$ if and only if $e(M) = 0$.

2.2. Basic Lemmas.

2.2.1. *Factorization of coverings of Seifert manifolds.* We show that any covering space of Seifert manifolds is a product of simpler coverings.

Lemma 2.3. *Assume that $G \leq S_n$ is a transitive group and $K \triangleleft G$. Assume that K has m orbits. Then there exist homomorphisms $q : G \rightarrow S_m$ and $\gamma : q^{-1}(St(1)) \rightarrow S_{n/m}$ such that $St_G(1) \subset q^{-1}(St(1))$, and $q(K) = 1$, and $\gamma(K)$ is transitive.*

Proof. This follows by noting that the orbits of K are a set of imprimitivity blocks for G . □

Corollary 2.4. *Let M be a Seifert manifold, and let $\varphi : \widetilde{M} \rightarrow M$ be an n -fold covering branched along fibers of M . Then there is a*

commutative diagram of coverings of Seifert manifolds branched along fibers

$$\begin{array}{ccc}
 \widetilde{M} & & \\
 \downarrow \varphi & \searrow \varphi_\gamma & \\
 & N & \\
 & \swarrow \varphi_q & \\
 M & &
 \end{array}$$

such that φ_q and φ_γ are m -fold and n/m -fold branched coverings, respectively, and, if $h \subset M$ is an ordinary fiber, then $\varphi_q^{-1}(h) = h_1 \sqcup \cdots \sqcup h_m$ with $\varphi_q| : h_i \rightarrow h$ a homeomorphism for $i = 1, \dots, m$, and $\varphi_\gamma^{-1}(\tilde{h})$ is connected for \tilde{h} any ordinary fiber of N .

Proof. Recall that the subgroup $\langle h \rangle \triangleleft \pi_1(M)$. Then $\omega(\langle h \rangle) \triangleleft \text{Image}(\omega)$ where $\omega : \pi_1(M - B_\varphi) \rightarrow S_n$ is the representation associated to φ , and B_φ is the branching of φ . Lemma 2.3 applies.

□

Remark 2.5. In the statement of Corollary 2.4, if ω_q , and ω_γ are the representations corresponding to φ_q , and φ_γ , respectively, the conclusions of Corollary 2.4 are that $\omega_q(h) = (1)$, and $\omega_\gamma(\tilde{h})$ is a cycle of order n/m for \tilde{h} an ordinary fiber of N .

2.2.2. Coverings of Seifert manifolds. Let $(F \times S^1, \{m_i\}_{i=1}^t)$ be a frame for the Seifert symbol $(Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$, and let $\omega : \pi_1(F \times S^1) \rightarrow S_n$ be a representation, and let $\varphi : \tilde{M} \rightarrow F \times S^1$ be the covering space associated to ω . If $\{\tilde{m}_j\}$ is a set of components of $\varphi^{-1}(\cup m_i)$, one for each component of $\varphi^{-1}(\bigcup_{i=1}^t (q_i \times S^1))$, then $(\tilde{M}, \{\tilde{m}_j\}_{j=1}^u)$ is a frame for some Seifert symbol $(Oo, \tilde{g}; B_1/A_1, \dots, B_u/A_u)$.

In this section we compute the numbers $\tilde{g}, u, A_1, B_1, \dots, A_u, B_u$ for some ‘generic’ representations $\omega : \pi_1(F \times S^1) \rightarrow S_n$ (see Remark 2.5). Proofs of these result can be found in [3].

Write $\varepsilon \in S_n$ for the n -cycle $\varepsilon = (1, 2, \dots, n)$.

Lemma 2.6. *Let $(F \times S^1, \{m_i\}_{i=1}^t)$ be a frame for $(Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$. Let $\omega : \pi_1(F \times S^1) \rightarrow S_n$ be a representation such that $\omega(h) = \varepsilon^s$ with $(n, s) = 1$, and $\omega(q_i) = \varepsilon^{r_i}$ for $i = 1, \dots, t$ with $\sum r_i = 0$. Let s^* be any integer such that $s^*s \equiv 1 \pmod{n}$.*

If $\varphi : \tilde{M} \rightarrow F \times S^1$ is the covering associated to ω and \tilde{m}_i is a component of $\varphi^{-1}(m_i)$, $i = 1, \dots, t$, then $(\tilde{M}, \{\tilde{m}_i\}_{i=1}^t)$ is a frame for $(Oo, g; B_1/A_1, \dots, B_t/A_t)$, where

$$A_i = \frac{n\alpha_i}{d_i}, \quad B_i = \frac{\beta_i + s^* r_i \alpha_i}{d_i}$$

and $d_i = (n, \beta_i + s^* r_i \alpha_i)$, $i = 1, \dots, t$.

□

Corollary 2.7. If $\sum r_i = 0$, and $(n, s) = 1$, and s^* is any integer such that $s^* s \equiv 1 \pmod{n}$, then $(Oo, g; B_1/A_1, \dots, B_t/A_t)$ is an n -fold branched covering of $(Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$, where

$$A_i = \frac{n\alpha_i}{d_i}, \quad B_i = \frac{\beta_i + s^* r_i \alpha_i}{d_i}$$

and $d_i = (n, \beta_i + s^* r_i \alpha_i)$, $i = 1, \dots, t$.

□

Lemma 2.8. Let $(F \times S^1, \{m_i\}_{i=1}^t)$ be a frame for $(Oo, g; \beta_1/1, \dots, \beta_u/1, \beta_{u+1}/\alpha_{u+1}, \dots, \beta_{u+t}/\alpha_{u+t})$ with $\alpha_j > 1$ for $j = u+1, \dots, u+t$. Let $\omega : \pi_1(F \times S^1) \rightarrow S_n$ be a transitive representation such that $\omega(h) = (1)$, and $\omega(q_i) = \varepsilon^{s_i}$ with $(n, s_i) = 1$ for $i = 1, \dots, u$, and $\omega(q_j)$ is a product of n_j disjoint cycles of order d_j and has k_j fixed points for $j = u+1, \dots, u+t$.

If $\varphi : \tilde{M} \rightarrow F \times S^1$ is the covering associated to ω and $\{\tilde{m}_j\}_j$ is a set of components of $\varphi^{-1}(\bigcup_{i=1}^{u+t} m_i)$, one for each component of $\varphi^{-1}(\bigcup_{i=1}^{u+t} (q_i \times S^1))$, then $(\tilde{M}, \{\tilde{m}_j\}_j)$ is a frame for

$$(Oo, \tilde{g}; \underbrace{n\beta_1/1, \dots, n\beta_u/1}_{k_{u+1} \text{ times}}, \underbrace{\beta_{u+1}/\alpha_{u+1}, \dots, \beta_{u+t}/\alpha_{u+t}}_{k_{u+t} \text{ times}}, \underbrace{\beta_{u+1}/\alpha'_{u+1}, \dots, \beta_{u+1}/\alpha'_{u+1}}_{n_{u+1} \text{ times}}, \dots, \underbrace{\beta_{u+t}/\alpha'_{u+t}, \dots, \beta_{u+t}/\alpha'_{u+t}}_{n_{u+t} \text{ times}}),$$

where $\alpha'_i = \alpha_i/d_i$ for $i = u+1, \dots, u+t$, and $\tilde{g} = (2 - n\chi(F) + u(n-1) + \sum n_i(d_i - 1))/2$.

□

Corollary 2.9. With the numbers d_i, n_i and k_i as in Lemma 2,

$$\begin{aligned}
& (Oo, \tilde{g}; \underbrace{n\beta_1/1, \dots, n\beta_u/1}_{k_{u+1} \text{ times}}, \underbrace{\beta_{u+1}/\alpha_{u+1}, \dots, \beta_{u+1}/\alpha_{u+1}}_{k_{u+t} \text{ times}}, \underbrace{\beta_{u+1}/\alpha'_{u+1}, \dots, \beta_{u+1}/\alpha'_{u+1}}_{n_{u+1} \text{ times}}, \dots, \\
& \underbrace{\beta_{u+t}/\alpha_{u+t}, \dots, \beta_{u+t}/\alpha_{u+t}}_{k_{u+t} \text{ times}}, \underbrace{\beta_{u+t}/\alpha'_{u+t}, \dots, \beta_{u+t}/\alpha'_{u+t}}_{n_{u+t} \text{ times}})
\end{aligned}$$

is an n -fold branched covering of

$$(Oo, g; \beta_1/1, \dots, \beta_u/1, \beta_{u+1}/\alpha_{u+1}, \dots, \beta_{u+t}/\alpha_{u+t})$$

where $\alpha_i > 1$, and $\alpha'_i = \alpha_i/d_i$ for $i = u+1, \dots, u+t$, and $\tilde{g} = (2 - n\chi(F) + u(n-1) + \sum n_i(d_i - 1))/2$.

□

Remark 2.10. In [3], the statements of Lemma 2.6 and Lemma 2.8 assume that the generators of the fundamental group of the orbit surface are sent to the identity by the representation. Although we only need that kind of covers in this paper, the hypothesis on the generators of the surface is not needed for the conclusions of the lemmas.

2.2.3. Fiberings of $S^2 \times S^1$.

Theorem 2.11. *Let M be a Seifert manifold with at most two exceptional fibers and base the 2-sphere. Then $M \cong S^2 \times S^1$ if and only if $e(M) = 0$.*

□

It follows that,

Corollary 2.12. *Let α, β be coprime integers such that $0 \leq \beta \leq \alpha/2$. Then the Seifert fiberings $(Oo, 0; \beta/\alpha, -\beta/\alpha)$ are all the Seifert fiberings of $S^2 \times S^1$.*

□

3. ABELIAN COVERINGS OF $S^2 \times S^1$.

Let $t_{\alpha, \beta}$ be an ordinary fiber of $(Oo, 0; \beta/\alpha, -\beta/\alpha)$. We compute the Seifert invariants of the Abelian coverings of $S^2 \times S^1$ branched along $t_{\alpha, \beta}$.

Write $M_0 = S^2 \times S^1 - t_{\alpha, \beta}$. Then we have the presentation

$$\begin{aligned}
\pi_1(M_0) = \langle q_0, q_1, q_2, h : q_1^\alpha h^\beta = 1, q_2^\alpha h^{-\beta} = 1, q_0 q_1 q_2 = 1, \\
[q_i, h] = 1, i = 0, 1, 2 \rangle.
\end{aligned}$$

Let $(F \times S^1, \{q_0, q_1^\alpha h^\beta, q_1^\alpha h^{-\beta}\})$ be a frame for $(Oo, 0; 0/1, \beta/\alpha, -\beta/\alpha)$. Then an Abelian representation $\omega : \pi_1(F \times S^1) \rightarrow S_n$ gives a branched covering of $(S^2 \times S^1, t_{\alpha, \beta})$ if and only if $\omega(q_1^\alpha h^\beta) = \omega(q_1^\alpha h^{-\beta}) = \omega(q_0 q_1 q_2) = (1)$. Since $H_1(M_0) \cong \mathbb{Z} \oplus \mathbb{Z}_\alpha$, the image of ω is cyclic or the sum of two cyclic groups. In Section 3.2 we describe these image groups.

3.1. The coverings. Let $(F \times S^1, \{q_0, q_1^\alpha h^\beta, q_1^\alpha h^{-\beta}\})$ be a frame for $(Oo, 0; 0/1, \beta/\alpha, -\beta/\alpha)$. Let $\omega : \pi_1(F \times S^1) \rightarrow S_n$ be a transitive representation with Abelian image and $\omega(q_1^\alpha h^\beta) = \omega(q_1^\alpha h^{-\beta}) = \omega(q_0 q_1 q_2) = (1)$, and let $\varphi : \tilde{M} \rightarrow (S^2 \times S^1, t_{\alpha, \beta})$ be the associated branched covering. As in Remark 2.5, $\varphi = \varphi_2 \circ \varphi_1$ with $\varphi_1 : \tilde{M} \rightarrow N$ and $\varphi_2 : N \rightarrow S^2 \times S^1$ branched coverings with corresponding representations ω_1 and ω_2 , and $\omega_1(\tilde{h})$ one cycle and $\omega_2(h) = (1)$.

We assume that $\omega(h) = (1)$ or that $\omega(h)$ is a cycle of order n .

Write $\omega(q_1) = \sigma_1$, and $\omega(q_2) = \sigma_2$.

3.1.1. First case: $\omega(h) = (1)$. Since $\omega(q_i)^\alpha = \sigma_i^\alpha = (1)$, then the orders $o(\sigma_i) = a_i$ satisfy $a_i | \alpha$.

Case 1.1: $a_1 = 1$; that is, $\sigma_1 = (1)$. Then σ_2 is an a_2 -cycle, and $n = a_2$, and $\omega(q_0) = \sigma_2^{-1}$. Write $\alpha' = \alpha/a_2$. By Lemma 2.8,

$$\tilde{M} = (Oo, 0; \underbrace{\beta/\alpha, \dots, \beta/\alpha}_{a_2 \text{ times}}, -\beta/\alpha'),$$

and $H_1(\tilde{M}) \cong \mathbb{Z} \oplus \mathbb{Z}_{\alpha'} \oplus \bigoplus_{j=1}^{a_2-2} \mathbb{Z}_\alpha$.

Case 1.2: $a_2 = 1$; that is, $\sigma_2 = (1)$. Then σ_1 is an a_1 -cycle, and $n = a_1$, and $\omega(q_0) = \sigma_1^{-1}$. Write $\alpha' = \alpha/a_1$. By Lemma 2.8,

$$\tilde{M} = (Oo, 0; \beta/\alpha', \underbrace{-\beta/\alpha, \dots, -\beta/\alpha}_{a_1 \text{ times}}),$$

and $H_1(\tilde{M}) \cong \mathbb{Z} \oplus \mathbb{Z}_{\alpha'} \oplus \bigoplus_{j=1}^{a_1-2} \mathbb{Z}_\alpha$.

Case 1.3: $a_1, a_2 > 1$. The pair σ_1, σ_2 has a 4-tuple (a_1, a_2, δ, i_0) , as in Section 3.2, with $\delta | (a_1, a_2)$, and $0 \leq i_0 < \delta$, and $(i_0, \delta) = 1$. Then $n = a_1 a_2 / \delta$, and

- σ_1 is a product of a_2/δ cycles of order a_1 .
- σ_2 is a product of a_1/δ cycles of order a_2 .
- Write $\mu = (a_1, a_2/\delta + a_1 i_0/\delta)$. Then the product $\sigma_1 \sigma_2$ is a product of $a_1 a_2 / (\mu \delta)$ cycles of order μ .

Write $\alpha_1 = \alpha/a_1$, and $\alpha_2 = \alpha/a_2$. By Lema 2.8,

$$\tilde{M} = (Oo, g; \underbrace{\beta/\alpha_1, \dots, \beta/\alpha_1}_{a_2/\delta \text{ times}}, \underbrace{-\beta/\alpha_2, \dots, -\beta/\alpha_2}_{a_1/\delta \text{ times}})$$

where

$$g = 1 + \frac{(a_1 - 1)(a_2 - 1) - 1}{2\delta} - \frac{\mu}{2},$$

and

$$H_1(\tilde{M}) \cong \mathbb{Z}^{2g+1} \oplus \bigoplus_{i=3}^{(a_1+a_2)/\delta} \mathbb{Z}_{d_i}$$

where

$$d_i = \begin{cases} \alpha/(a_1, a_2), & \text{if } i \leq a_1/\delta, a_2/\delta, \\ \alpha_1, & \text{if } a_1/\delta < i \leq a_2/\delta, \\ \alpha_2, & \text{if } a_2/\delta < i \leq a_1/\delta, \\ \alpha/[a_1, a_2], & \text{if } a_1/\delta, a_2/\delta < i. \end{cases}$$

Remark 3.1. The genus g of the orbit surface of \tilde{M} is zero if and only if either $\delta = a_1 = a_2$, or $\delta = 1$ and $a_1 = a_2 = 2$.

Indeed if $g = 0$, then

$$2\delta + (a_1 - 1)(a_2 - 1) - 1 = \delta\mu \leq a_1,$$

and then

$$a_2 - 1 \leq \frac{a_1 - 2}{a_1 - 1}\delta + \frac{1}{a_1 - 1} < \delta + 1.$$

Therefore $a_2 \leq \delta + 1$. Since $\delta|a_2$ and $\delta \leq a_2$, it follows that either $a_2 = \delta$, or $\delta = 1$ and $a_2 = 2$.

Since the pair σ_2, σ_1 in reverse order has a 4-tuple (a_2, a_1, δ, j_0) , as in Section 3.2, with $\delta|(a_1, a_2)$, and $0 \leq j_0 < \delta$, and $(j_0, \delta) = 1$, we see that $\mu = (a_2, a_1/\delta + a_2j_0/\delta)$, and it follows, as above, that either $a_1 = \delta$, or $\delta = 1$ and $a_1 = 2$.

Then, either

- $\delta = a_1 = a_2$, and φ is an unbranched a_1 -fold cyclic covering of $S^2 \times S^1$, or
- $\delta = 1$ and $a_1 = a_2 = 2$. In this case, up to conjugation, $\sigma_1 = (1, 2)(3, 4)$, and $\sigma_2 = (1, 3)(2, 4)$, and, if $\alpha' = \alpha/2$, then $\tilde{M} = (Oo, 0; \beta/\alpha', \beta/\alpha', -\beta/\alpha', -\beta/\alpha')$.

3.1.2. *Second case:* $\omega(h) = (1, 2, \dots, n) = \varepsilon \in S_n$. Write $\omega(q_1) = \varepsilon^{r_1}$ and $\omega(q_2) = \varepsilon^{r_2}$. Since $\omega(q_1^\alpha h^\beta) = \varepsilon^{\alpha r_1 + \beta} = (1)$, and $\omega(q_2^\alpha h^{-\beta}) = \varepsilon^{\alpha r_2 - \beta} = (1)$, it follows that

$$r_1\alpha + \beta \equiv 0 \pmod{n}$$

and

$$r_2\alpha - \beta \equiv 0 \pmod{n}.$$

Since $(\alpha, \beta) = 1$, this two equations have solution if and only if $(n, \alpha) = 1$. Let α^* be an integer such that $\alpha^*\alpha \equiv 1 \pmod{n}$, and write $r_1 = -\alpha^*\beta$ and $r_2 = \alpha^*\beta$; then $r_1 + r_2 = 0$, and r_1, r_2 are solutions of the system, if $(n, \alpha) = 1$. Say $r_1\alpha + \beta = kn$; then $r_2\alpha - \beta = -kn$.

Since $\omega(q_0) = \omega((q_1q_2)^{-1}) = \varepsilon^{-(r_1+r_2)} = (1)$, then φ is an unbranched cyclic covering of $S^2 \times S^1$.

Write $A_1 = n\alpha/d_1$, $B_1 = (\beta + r_1\alpha)/d_1$, and $A_2 = n\alpha/d_2$, $B_2 = (-\beta + r_2\alpha)/d_2$ where $d_1 = (n, \beta + r_1\alpha)$ and $d_2 = (n, -\beta + r_2\alpha)$. Then, by Lemma 2.6, $\tilde{M} = (Oo, 0; B_1/A_1, B_2/A_2)$. Notice that $d_1 = (n, \beta + r_1\alpha) = (n, kn) = n$. Then $A_1 = \alpha$, and $B_1 = k$. Similarly $A_2 = \alpha$, and $B_2 = -k$. Then

$$\tilde{M} = (Oo, 0; k/\alpha, -k/\alpha).$$

and $\tilde{M} \cong S^2 \times S^1$.

Theorem 3.2. *Let α, β be coprime integers with $0 \leq \beta \leq \alpha/2$, and let $\omega : \pi_1(S^2 \times S^1 - t_{\alpha, \beta}) \rightarrow S_n$ be a transitive representation with Abelian image. Let $\varphi : \tilde{M} \rightarrow (S^2 \times S^1, t_{\alpha, \beta})$ be the branched covering associated with ω . Let h be an ordinary fiber of $S^2 \times S^1 - t_{\alpha, \beta}$, and write the order $o(\omega(q_i)) = a_i$ where we write $a_i = 1$ if $\omega(q_i) = (1)$. Also write $\alpha_i = \alpha/a_i$, $i = 1, 2$.*

- (1) *If $\omega(h) = (1)$, then $n = a_1a_2/\delta$ for some divisor $\delta | (a_1, a_2)$, and*

$$\tilde{M} = (Oo, g; \underbrace{\beta/\alpha_1, \dots, \beta/\alpha_1}_{a_2/\delta \text{ times}}, \underbrace{-\beta/\alpha_2, \dots, -\beta/\alpha_2}_{a_1/\delta \text{ times}})$$

where

$$g = 1 + \frac{(a_1 - 1)(a_2 - 1) - 1}{2\delta} - \frac{\mu}{2},$$

and $\mu = (a_1, a_2/\delta + a_1i_0/\delta)$ for some i_0 such that $(i_0, \delta) = 1$, and $0 \leq i_0 < \delta$. Also

$$H_1(\tilde{M}) \cong \mathbb{Z}^{2g+1} \oplus \bigoplus_{i=3}^{(a_1+a_2)/\delta} \mathbb{Z}_{d_i}$$

where

$$d_i = \begin{cases} \alpha/(a_1, a_2), & \text{if } i \leq a_1/\delta, a_2/\delta, \\ \alpha_1, & \text{if } a_1/\delta < i \leq a_2/\delta, \\ \alpha_2, & \text{if } a_2/\delta < i \leq a_1/\delta, \\ \alpha/[a_1, a_2], & \text{if } a_1/\delta, a_2/\delta < i. \end{cases}$$

If $a_1, a_2 > 1$, then $g = 0$ if and only if either $\delta = a_1 = a_2$, or $\delta = 1$ and $a_1 = a_2 = 2$. If $g = 0$ and $\delta = a_1 = a_2$, then $\tilde{M} \cong S^2 \times S^1$ and φ is an unbranched cyclic covering.

- (2) If $\omega(h) = (1, 2, \dots, n)$, then $(n, \alpha) = 1$, and φ is an unbranched covering space, and

$$\tilde{M} = (Oo, 0; k/\alpha, -k/\alpha) \cong S^2 \times S^1$$

where $k = (-\beta\alpha^*\alpha + \beta)/n$, and $\alpha^*\alpha \equiv 1 \pmod{n}$.

Corollary 3.3. Let $\varphi : \tilde{M} \rightarrow (S^2 \times S^1, t_{\alpha, \beta})$ be an n -fold Abelian branched covering. If $H_1(\tilde{M}) \cong \mathbb{Z}$, then $\tilde{M} \cong S^2 \times S^1$ and φ is an unbranched cyclic covering space.

3.2. Abelian permutation groups generated by two elements.

Let $\sigma_1, \sigma_2 \in S_n$ be a pair of permutations such that the group $G = \langle \sigma_1, \sigma_2 \rangle$ is Abelian and transitive.

We associate a 4-tuple of integers (a_1, a_2, δ, i_0) to the pair σ_1, σ_2 .

Assume that the order $o(\sigma_i) = a_i$ where we write $a_i = 1$, if $\sigma_i = (1)$ ($i = 1, 2$). Then $n = o(G) = a_1 a_2 / \delta$ for some divisor $\delta | (a_1, a_2)$.

Let T be a torus with a CW structure with one vertex v , two 1-cells, γ_1, γ_2 and one 2-cell d . Since σ_1 and σ_2 commute, the assignment $\omega : \pi_1(T) \rightarrow S_n$ such that $\omega(\gamma_i) = \sigma_i$ defines a representation. Let $\varphi : \tilde{T} \rightarrow T$ be the covering space associated to ω . Then \tilde{T} admits a CW structure with cells the liftings of the cells of T .

The 1-skeleton of this CW structure of \tilde{T} gives a portrait of σ_1 and σ_2 : Draw a grid of a_2/δ horizontal, parallel lines with $a_1 + 1$ equally spaced vertices each, so that, after identifying the ends of a line, we obtain a circle with a_1 vertices, and complete the grid with a_1 vertical lines with $a_2/\delta + 1$ vertices. Now the top and the bottom of the grid are identified in such a way that the vertical lines become a_1/δ circles with a_2 vertices each: that is, δ equally spaced vertical lines become a circle.

Then for some number i_0 such that $(\delta, i_0) = 1$, and $0 \leq i_0 < \delta$, after identifying the sides of the grid with the identity, the top of the grid

is identified with the bottom with a twist that identifies the top of the first vertical line with the bottom of the $(i_0\delta)$ -th vertical line.

The pair σ_1, σ_2 can be recovered from the 4-tuple (a_1, a_2, δ, i_0) , up to conjugation. Notice that if we reverse the order of σ_1, σ_2 , the pair σ_2, σ_1 has a 4-tuple of the form (a_1, a_2, δ, j_0) with $(\delta, j_0) = 1$, and $0 \leq j_0 < \delta$ (usually $i_0 \neq j_0$).

The product $\sigma_1\sigma_2$ is drawn on \tilde{T} as the union of the liftings of the diagonal of the 2-cell d of T which is homotopic to $\gamma_1\gamma_2$. Therefore the order

$$o(\sigma_1\sigma_2) = \frac{a_1a_2/\delta}{(a_1, (i_0a_1 + a_2)/\delta)}.$$

4. UNIVERSALS.

Write $|X|$ for the number of components of the space X . In this section we fix integers α, β such that $(\alpha, \beta) = 1$, $0 \leq \beta \leq \alpha/2$. We write $t_{\alpha, \beta} \subset S^2 \times S^1$ for the torus knot which is an ordinary fiber of $(Oo, 0; \beta/\alpha, -\beta/\alpha)$.

Lemma 4.1. *There exists $\varphi : (Oo, 0;) \rightarrow (S^2 \times S^1, t_{\alpha, \beta})$ a 2α -fold branched covering such that $|\varphi^{-1}(t_{\alpha, \beta})| = 2\alpha - 2$.*

Proof. Let $(F \times S^1, \{m_0, m_1, m_2\})$ be a frame for $(Oo, 0; 0/1, \beta/\alpha, -\beta/\alpha)$, and let $\omega : F \times S^1 \rightarrow S_{2\alpha}$ be the representation such that

$$\begin{aligned} \omega(h) &= (1) \\ \omega(q_0) &= (\alpha - 1, 2\alpha - 1)(\alpha, 2\alpha) \\ \omega(q_1) &= (1, 2, \dots, \alpha - 2, \alpha - 1, 2\alpha)(\alpha, \alpha + 1, \alpha + 2, \dots, 2\alpha - 2, 2\alpha - 1) \\ \omega(q_2) &= (2\alpha, 2\alpha - 1, \dots, \alpha + 2, \alpha + 1)(\alpha, \alpha - 1, \dots, 2, 1) \end{aligned}$$

Notice that $\omega(m_1) = \omega(q_1^\alpha h^\beta) = (1)$, and $\omega(m_2) = \omega(q_1^\alpha h^{-\beta}) = (1)$, and $\omega(q_0 q_1 q_2) = (1)$. By Lemma 2.8, the completion of the covering associated with ω is

$$\varphi : (Oo, 0; \frac{\beta}{1}, \frac{\beta}{1}, \frac{-\beta}{1}, \frac{-\beta}{1}) \rightarrow (Oo, 0; \frac{0}{1}, \frac{\beta}{\alpha}, \frac{-\beta}{\alpha}),$$

and is branched along the $0/1$ -fiber, that is, is branched along $t_{\alpha, \beta}$. Since $\omega(q_0)$ has $2\alpha - 4$ fixed points and two 2-cycles, therefore $|\varphi^{-1}(t_{\alpha, \beta})| = 2\alpha - 2$. \square

Lemma 4.2. *If $k > 0$ and $\alpha \geq 3$, then there exists $\varphi : (Oo, 0;) \rightarrow (S^2 \times S^1, t_{\alpha, \beta})$ a branched covering such that $|\varphi^{-1}(t_{\alpha, \beta})| \geq k$.*

Proof. By previous lemma, there is $\psi : (Oo, 0;) \rightarrow (S^2 \times S^1, t_{\alpha, \beta})$ with $|\psi^{-1}(t_{\alpha, \beta})| = 2\alpha - 2 \geq 4$. Let $(F \times S^1, \{m_1, m_2\})$ be a frame for $(Oo, 0; 0/1, 0/1)$ where the $0/1$ -fibers e_1, e_2 are contained in $\psi^{-1}(t_{\alpha, \beta})$. Let $\omega : F \times S^1 \rightarrow S_k$ be the representation such that $\omega(h) = (1)$, $\omega(q_1) = \varepsilon$, and $\omega(q_2) = \varepsilon^{-1}$. where $\varepsilon = (1, 2, \dots, k) \in S_k$. By Lemma 2.8, the completion of the covering associated with ω is

$$\varphi : (Oo, 0; \frac{0}{1}, \frac{0}{1}) \rightarrow (Oo, 0;)$$

and is branched along $e_1 \cup e_2$. Let $e_3 \subset \psi^{-1}(t_{\alpha, \beta})$ be a fiber different from e_1 and e_2 . Then $|\varphi^{-1}(e_3)| = k$, and the lemma follows. \square

Lemma 4.3. *Let $g \geq 0$, and $k \geq 2g+2$ be integers, and assume $\alpha \geq 3$. Then there exists a branched covering $\varphi : (Oo, g;) \rightarrow (S^2 \times S^1, t_{\alpha, \beta})$ such that $|\varphi^{-1}(t_{\alpha, \beta})| \geq k$.*

Proof. By previous lemma, there is $\varphi : (Oo, 0;) \rightarrow (S^2 \times S^1, t_{\alpha, \beta})$ a branched covering such that $|\varphi^{-1}(t_{\alpha, \beta})| \geq k \geq 2g+2$. Then there are $2g+2$ different fibers $e_1, e_2, \dots, e_{2g+2} \subset |\varphi^{-1}(t_{\alpha, \beta})|$. The double cyclic covering of $(Oo, 0;)$ branched along $e_1 \cup e_2 \cup \dots \cup e_{2g+2}$ is $(Oo, g;)$ (the double branched covering, in terms of a representation ω of a frame for $(Oo, 0; 0/1, \dots, 0/1)$ into S_2 , is $\omega(h) = (1)$, and $\omega(q_i) = (1, 2)$ for $i = 1, 2, \dots, 2g+2$). \square

Lemma 4.4. *Let $g \geq 0$ and $k > 0$ be integers, and assume $\alpha \geq 3$. If b/a is a reduced rational number, then there exists a branched covering $\varphi : (Oo, g; b/a, -b/a) \rightarrow (S^2 \times S^1, t_{\alpha, \beta})$ such that $|\varphi^{-1}(t_{\alpha, \beta})| \geq k$ and $\varphi^{-1}(t_{\alpha, \beta})$ contains the fibers of the ratios of $(Oo, g; b/a, -b/a)$.*

Proof. By previous lemma, there is a branched covering $\psi : (Oo, g;) \rightarrow (S^2 \times S^1, t_{\alpha, \beta})$ with $|\psi^{-1}(t_{\alpha, \beta})| \geq k$. Let $(F \times S^1, \{m_0, m_1, m_2, m_3\})$ be a frame for $(Oo, g; 0/1, 0/1, b/1, -b/1)$, and let $\omega : F \times S^1 \rightarrow S_a$ be the representation such that

$$\begin{aligned} \omega(h) &= \varepsilon^{b^*} \\ \omega(q_0) &= \omega(q_2) = \varepsilon^{-1} \\ \omega(q_1) &= \omega(q_3) = \varepsilon \end{aligned}$$

where $b^*b \equiv 1 \pmod{a}$, and $\varepsilon = (1, 2, \dots, a) \in S_a$. By Lemma 2.8, the completion of the covering associated with ω is

$$\varphi : (Oo, g; \frac{b}{a}, \frac{-b}{a}, \frac{0}{1}, \frac{0}{1}) \rightarrow (Oo, g;)$$

and is branched along the fibers of the ratios of $(Oo, g; 0/1, 0/1, b/1, -b/1)$. \square

Theorem 4.5. *Assume that $\alpha \geq 3$. If $\sum_{i=1}^t \beta_i/\alpha_i = 0$, then there exists a branched covering $\varphi : (Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t) \rightarrow (S^2 \times S^1, t_{\alpha, \beta})$ such that $\varphi^{-1}(t_{\alpha, \beta})$ contains the fibers of the ratios of $(Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$ and some extra 0/1 fibers.*

Proof. Induction on t . Assume that $t = 2$. If $\alpha_1 = 1$, by Lemma 4.3, $(Oo, g; \beta_1/1, -\beta_1/1)$ is a branched covering of $(S^2 \times S^1, t_{\alpha, \beta})$; otherwise, by Lemma 4.4, $(Oo, g; \beta_1/\alpha_1, -\beta_1/\alpha_1)$ is a branched covering of $(S^2 \times S^1, t_{\alpha, \beta})$ as needed.

Assume that the theorem is true for $t \geq 2$, that is, if the sum of the ratios $\sum_{i=1}^t \beta_i/\alpha_i = 0$, then the manifold $(Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$ is a branched covering of $(S^2 \times S^1, t_{\alpha, \beta})$ with a branched covering φ such that $\varphi^{-1}(t_{\alpha, \beta})$ contains the fibers of the ratios of the manifold $(Oo, g; \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$, plus some extra 0/1 fibers. Assume now that $\sum_{i=1}^t \beta_i/\alpha_i + u/v = 0$.

Let $(F \times S^1, \{m_0, m_1, \dots, m_t\})$ be a frame for the manifold $(Oo, g; u/1, b_1/a_1, \dots, b_t/a_t)$ where the b_i/a_i is the reduced ratio $v\beta_i/\alpha_i$; that is, $b_i = v\beta_i/(v, \alpha_i)$, and $a_i = \alpha_i/(v, \alpha_i)$ for $i = 1, \dots, t$. Let $\omega : \pi_1(F \times S^1) \rightarrow S_v$ be the representation such that $\omega(h) = (1, 2, \dots, v)$, and $\omega(q_i) = (1)$. Here we have $s = 1$ and $r_i = 0$. Then, by Lemma 2.6, the completion of the covering associated with ω is

$$\varphi : (Oo, g, \frac{B}{A}, \frac{B_1}{A_1}, \dots, \frac{B_t}{A_t}) \rightarrow (Oo, g; \frac{\beta}{1}, \frac{b_1}{a_1}, \dots, \frac{b_t}{a_t})$$

where $d = (v, u) = 1$, and $B = u/d = u$, and $A = v/d = v$, and for $i \in \{1, \dots, t\}$, $d_i = (v, b_i) = (v, v\beta_i/(v, \alpha_i)) = v/(v, \alpha_i)$, and $B_i = b_i/d_i = (v\beta_i/(v, \alpha_i))/(v/(v, \alpha_i)) = \beta_i$, and $A_i = va_i/d_i = (v\alpha_i/(v, \alpha_i))/(v/(v, \alpha_i)) = \alpha_i$. That is, $(Oo, g; u/v, \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$ is a branched covering of $(Oo, g; u/1, b_1/a_1, \dots, b_t/a_t)$ with branching along the fibers of the ratios.

Since $u/1 + \sum_{i=1}^t b_i/a_i = v(u/v + \sum_{i=1}^t \beta_i/\alpha_i) = 0$, and $(Oo, g; u/1, b_1/a_1, \dots, b_t/a_t)$ is isomorphic to $(Oo, g; (ua_1 + b_1)/a_1, b_2/a_2, \dots, b_t/a_t)$, by induction, the manifold $(Oo, g; u/1, b_1/a_1, \dots, b_t/a_t)$ is a branched covering of $(S^2 \times S^1, t_{\alpha, \beta})$ with a branched covering ψ such that $\psi^{-1}(t_{\alpha, \beta})$ contains the fibers of the ratios of $(Oo, g; u/1, b_1/a_1, \dots, b_t/a_t)$ plus some extra 0/1 fibers. We conclude that $(Oo, g; u/v, \beta_1/\alpha_1, \dots, \beta_t/\alpha_t)$ is also a branched covering of $(S^2 \times S^1, t_{\alpha, \beta})$ as required.

□

Corollary 4.6. *Let M be an orientable Seifert manifold with orientable basis and $e(M) = 0$, and assume that $\alpha \geq 3$. Then M is a branched covering of $(S^2 \times S^1, t_{\alpha, \beta})$.*

4.1. A non-trivial non-universal torus knot in $S^2 \times S^1$.

Theorem 4.7. *If M is a branched covering of $(S^2 \times S^1, t_{2,1})$, then $M \cong S^2 \times S^1$.*

Proof. Let $\varphi : M \rightarrow (S^2 \times S^1, t_{2,1})$ be an n -fold branched covering, and write $F = S^2 \times \{1\}$. Then F is a non-separating 2-sphere generating $H_2(S^2 \times S^1)$, and $|F \cap t_{2,1}| = 2$; thus $\varphi^{-1}(F)$ is a disjoint union of 2-spheres. Now an orientation on F induces an orientation on $\varphi^{-1}(F)$ through φ , and with this orientation, $\varphi^{-1}(F)$ defines a class in $H_2(M)$. since $\varphi| : \varphi^{-1}(F) \rightarrow F$ is an n -fold branched covering, then $\varphi_*[\varphi^{-1}(F)] = n[F] \neq 0$. In particular $[\varphi^{-1}(F)] \in H_2(M) - \{0\}$, and, therefore, some component of $\varphi^{-1}(F)$ is non-separating. Then M is an orientable Seifert manifold with a non-separating 2-sphere. We conclude that $M \cong S^2 \times S^1$. □

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